Exercise 9

$$\frac{\partial^2}{\partial z^2} E_3(z) + \frac{\omega_3^2}{c^2} \cdot \epsilon(\omega_3) \cdot E_3(z) = -\frac{\omega_3^2}{\epsilon_0 c^2} P_3(z)$$

$$E_3 = A_3(z) e^{ik_3 z} \cdot e^{-i\omega_3 t} + c.c$$

$$P_3(z) = 2\epsilon_0 \chi^{(2)} A_1 A_2 e^{i(k_1+k_2)z} e^{-i\omega_3 t} + c.c$$

$$\frac{-\omega_3^2}{\epsilon_0 c^2} P_3(z) = -\frac{2\omega_3^2 \cdot \chi^{(2)} A_1 A_2}{c^2} \cdot e^{i(k_1+k_2)z} \cdot e^{-i\omega_3 t} + c.c.$$

$$\frac{\omega_3^2}{c^2} \epsilon(\omega_3) \cdot E_3(z) = \frac{\epsilon(\omega_3)\omega_3^2 \cdot A_3(z)}{c^2} \cdot e^{ik_3 z} \cdot e^{-i\omega_3 t} + c.c.$$

$$k = \frac{n\omega}{c} \qquad k^2 = \frac{\overset{\epsilon}{\updownarrow}}{} \frac{n^2\omega^2}{c^2}$$

$$= \underline{k_3^2 \cdot A_3(z)} e^{ik_3 z} \cdot e^{-i\omega_3 t} + c.c.$$

$$\frac{\partial^2}{\partial z^2} E_3(z) = \frac{\partial^2}{\partial z^2}\left[A_3(z) e^{ik_3 z}\right] e^{-i\omega_3 t}$$

$$\frac{\partial}{\partial z}\left(\left[\frac{\partial A_3}{\partial z} + A_3(z) ik_3\right] e^{ik_3 z}\right) e^{-i\omega_3 t}$$

$$\left[\frac{\partial^2 A_3}{\partial z^2} + 2ik_3 \frac{\partial A_3}{\partial z} - \underline{k_3^2 A_3}\right] e^{ik_3 z} \cdot e^{-i\omega_3 t}$$

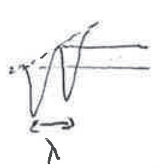
left side sum = right side

$$\left[\underset{\downarrow\, 0}{\frac{\partial^3 A_3}{\partial z^2}} + 2ik_3 \frac{\partial A_3}{\partial z}\right] e^{ik_3 z} \cdot e^{-i\omega_3 t} + c.c. = -2\chi^{(2)} \frac{\omega_3^2}{c^2} A_1 A_2 \cdot e^{i(k_1+k_2)z} \cdot e^{-i\omega_3 t} + c.c.$$

Using the slowly varying wave approximation

$\frac{dA_3}{dz} \cdot k_3$ = change of A_3 over a distance λ_3

$\frac{d^2A_3}{dz^2}$ ~ 0 because its slope changes very little



$\frac{dA_3}{dz}$ per λ

$$\frac{d^2A_3}{dz^2} < \frac{dA_3}{dz}$$

Next we drop c.c and $e^{-i\omega_3 t}$ from each side:

$$\frac{dA_3}{dz} = \frac{i\chi^{(2)}\omega_3^2}{k_3 c^2} A_1 A_2 \, e^{i(k_1+k_2-k_3)z}$$

$\Delta K = k_1 + k_2 - k_3$

and $\frac{-1}{i} \cdot \frac{i}{i} = i$

Following the same procedure for P_1 and P_2

$$P_1 = 2\epsilon_0 \chi^{(2)} A_3 A_2^* \, e^{i(k_3-k_2)z} \cdot e^{-i\omega_1 t}$$

$\omega_1 = \omega_3 - \omega_2$

$$P_2 = 2\epsilon_0 \chi^{(2)} A_3 A_1^* \, e^{i(k_3-k_1)z} \cdot e^{-i\omega_2 t}$$

$\omega_2 = \omega_3 - \omega_1$

and $E_1 = A_1(z) e^{ik_1 z - \omega_1 t}$

$E_2 = A_2(z) e^{ik_2 z - \omega_2 t}$

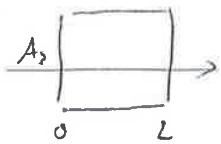
We obtain:

$$\frac{dA_1}{dz} = \frac{i\chi^{(2)}\omega_1^2}{k_1 c^2} \cdot A_3 A_2^* \, e^{-i\Delta k z}$$

$$\frac{dA_2}{dz} = \frac{i\chi^{(2)}\omega_2^2}{k_2 c^2} A_3 A_1^* \, e^{-i\Delta k z}$$

To obtain the intensity of the emitted field we use:

$I_i = 2 n_i \epsilon_0 c \, |A_i|^2$; for A_3 we have



$$A_3(L) = \cancel{4}\int_0^L d(A_3(z))$$

$$= \int_0^L \frac{i\chi^{(2)}\omega_3^2 A_1 A_2}{k_3 c^2}\, e^{i\Delta k z}\, dz = \frac{i\chi^{(2)}\omega_3^2 \cdot A_1 A_2}{k_3 c^2}\left[\frac{e^{i\Delta k L} - 1}{i\Delta k}\right]$$

and $I_3 = 2 n_3 \epsilon_0 c \, |A_3|^2$

$$= \underbrace{\frac{2 n_3 \epsilon_0 c \, |\chi^{(2)}|^2 \omega_3^4 \, |A_1|^2 |A_2|^2}{k_3^2 c^4}}_{2} \; \underbrace{\left|\frac{e^{i\Delta k L} - 1}{i\Delta k}\right|^2}_{1}$$

1: $\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2} \; ; \; \cos 2x = 1 - 2\sin^2 x \; ; \; 2\sin^2 x = 1 - \cos 2x}$ $\qquad \boxed{\frac{\sin^2 x}{x^2} = \mathrm{sinc}^2 x}$

$2(1 - 2\sin^2(\frac{\Delta k L}{2}))$

$$\frac{e^{i\Delta k L} - 1}{i\Delta k} \cdot \frac{e^{-i\Delta k L} - 1}{-i\Delta k} = \frac{1 - e^{-i\Delta k L} - e^{i\Delta k L} + 1}{\Delta k^2} = \frac{2 - 2\cos(\Delta k L)}{\Delta k^2} = 4\sin^2\left(\frac{\Delta k L}{2}\right) \cdot \frac{1}{\Delta k^2} \cdot \frac{L^2}{L^2}$$

$$\frac{4 \sin^2\left(\frac{\Delta k L}{2}\right)}{\Delta k^2} = \frac{\sin^2\left(\frac{\Delta k L}{2}\right)}{\left(\frac{\Delta k \cdot L}{2}\right)^2} \cdot L^2 = \underline{L^2 \cdot \mathrm{sinc}^2\left(\frac{\Delta k L}{2}\right)} \qquad \text{unit: } m^2$$

$\cancel{= L^2 \frac{\sin^2(\Delta k L)}{(\Delta k L/2)^2} = L^2 \sin^2(\frac{\Delta k L}{2})}$

2: $$\frac{2 n_3 \epsilon_0 |\chi^{(2)}|^2 \omega_3^4}{k_3^2 \cdot c^3} \frac{I_1}{2 n_1 \epsilon_0 c} \cdot \frac{I_2}{2 n_2 \epsilon_0 c} = \cdots\!\!> \frac{n_3 \omega_3^4 \epsilon_0 I_1 \cdot I_2 \cdot |\chi|^2}{2 k_3^2 \cdot n_1 \cdot n_2 \cdot \epsilon_0^2 \cdot c^5} \qquad \frac{\omega_3 n_3}{c} = k_3$$

$$= \frac{n_3 \omega_3^4 \epsilon_0 \cdot I_1 \cdot I_2 \cdot |\chi^{(2)}|^2}{2 \omega_3^2 \cdot n_3^2 \cdot c^{-2} \cdot n_1 \cdot n_2 \cdot \epsilon_0^2 \cdot c^5} = \underline{\underline{\frac{\omega_3^2 \, |\chi^{(2)}|^2 I_1 \cdot I_2}{2 \cdot n_3 \cdot n_2 \cdot n_1 \cdot \epsilon_0 \cdot c^3}}}$$

units: $$\frac{J^2 \cdot m^{-4} \cdot s^{-2} \cdot s^{-2} \cdot m^2 \cdot V^{-2}}{\underbrace{J \cdot V^{-2} \cdot m^{-1}}_{\epsilon_0} \cdot m^3 \cdot s^{-3}} = \frac{J^2 \cdot V^{-2} m^{-2} \cdot s^{-4}}{J \cdot V^{-2} \cdot m^2 \cdot s^{-3}} = J \cdot m^{-4} s^{-1}$$

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Thus I_3 becomes:

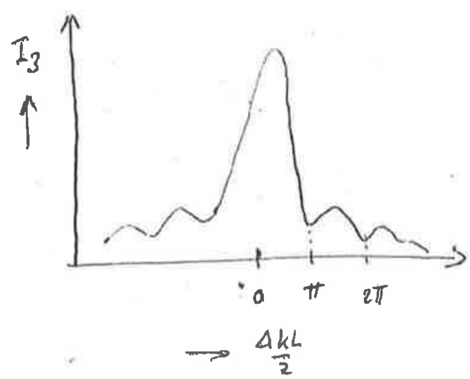
$$\boxed{I_3 = \frac{\omega_3^2 \cdot |\chi^{(2)}|^2\, I_1 \cdot I_2 \cdot L^2 \cdot \mathrm{sinc}^2\left(\frac{\Delta k L}{2}\right)}{2\epsilon_0 c^3 n_3 n_2 \cdot n_1}}$$

unit: $J \cdot m^{-4} \cdot s^{-1} \cdot m^2 = J/m^2 \cdot s$.

Note that this is different from Eq. 2.2.19) $|\chi^{(2)}|^2 = |2d_{eff}|^2 = 4d_{eff}^2$ would give $2d_{eff}^2$ instead of $8d_{eff}^2$; as well there is a $\frac{1}{c}$ term less.

The spatial dependence is included in the part

$I_3(\Delta k L) = L^2 \cdot \mathrm{sinc}^2\left(\frac{\Delta k L}{2}\right)$. This has the form:

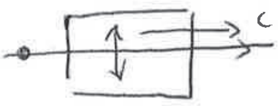


if $L = \frac{2\pi}{\Delta k}$ $I_3 \to 0$.

A typical length for modulation of the phasematching condition is the coherence length which can be arbitrarily written as

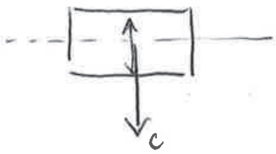
$L_{coh} = 2/\Delta k \Rightarrow I_3 \cong L_{coh}^2 \cdot \mathrm{sinc}^2(1)$

9 d.

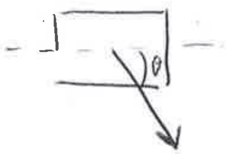


$\theta = 0$
$n_e(0) = n_o$

Normal dispersion $n(\omega) < n(2\omega)$ therefore 2ω should experience $\tilde{n}_e(\theta)$ which is smaller than n_o.
The polarization of both k_ω and $k_{2\omega}$ needs to be orthogonal



$n_e(90) = \tilde{n}_e$



$n_e(45)$: $\dfrac{1}{n_e^2(45)} = \dfrac{1}{2n_o^2} + \dfrac{1}{2n_e^2}$

$$\tilde{n}_e(45) = \sqrt{\frac{2n_o^2\tilde{n}_e^2}{n_o^2 + \tilde{n}_e^2}}$$

e. phase matching condition: $n(2\omega) = \cancel{2}\, n(\omega)$

$n_e(2\omega, \theta) = n_o(\omega)$

$\dfrac{1}{n_e^2(2\omega,\theta)} = \dfrac{1}{n_o^2(\omega)}$ first term is given in exercise:

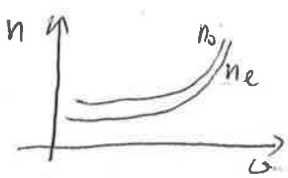
$$\frac{\sin^2\theta}{\tilde{n}_e^2(2\omega)} + \frac{\cos^2\theta}{n_o^2(2\omega)} = \frac{1}{n_o^2(\omega)} \qquad \cos^2\theta = 1 - \sin^2\theta$$

$$\frac{\sin^2\theta}{\tilde{n}_e^2(2\omega)} + \frac{1}{n_o^2(2\omega)} - \frac{\sin^2\theta}{n_o^2(2\omega)} = \frac{1}{n_o^2(\omega)} \dashrightarrow \sin^2\theta\left\{\frac{1}{\tilde{n}_e^2(2\omega)} - \frac{1}{n_o^2(2\omega)}\right\} = \frac{1}{n_o^2(\omega)} - \frac{1}{n_o^2(2\omega)}$$

$$\sin^{*}\theta = \sqrt{\frac{\frac{1}{n_o^2(\omega)} - \frac{1}{n_o^2(2\omega)}}{\frac{1}{\tilde{n}_e^2(2\omega)} - \frac{1}{n_o^2(2\omega)}}}$$

} normal dispersion in x-tal
} birefringence

f. For a negative uniaxial crystal we typically have

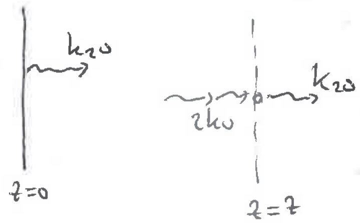


$\omega\uparrow$ n_o, n_e become closer; the difference reduces with ω so that the birefringence will become too small and $\sin\theta > 1$. → no possible phase matching
Likewise $\dfrac{1}{n_o^2(\omega)} - \dfrac{1}{n_o^2(2\omega)}$ can become too small.
Effects of absorbtion ~~that~~ also complicate effects as it enhances the amount of dispersion.

9g. Phasematching for SHG in transmission and reflection

Transmission:

H_2O



phase of light   0

$2k_\omega z$ SH generated at $z=z$

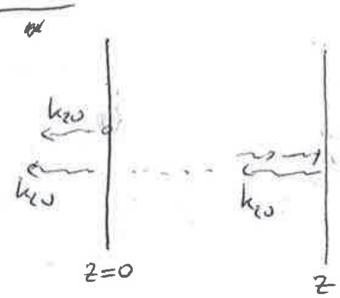
$z \cdot k_{2\omega}$ generated at $z=0$ travels to plane z

$$z(2k_\omega - k_{2\omega})$$

$$|z(2k_\omega - k_{2\omega})| = \pi \qquad k = \frac{\omega n}{c} = \frac{2\pi n}{\lambda}$$

$$z\left[\frac{2\pi \cdot 1.34}{500} - \frac{4\pi \cdot 1.33}{1000}\right] = \pi \Rightarrow z = 26{,}9\ \mu m$$

Reflection

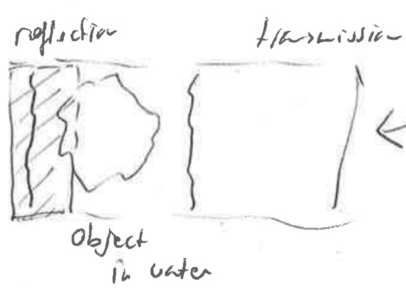


generated at $z=0$ 0

generated at z $2zk_\omega + k_{2\omega}$: $z\left(\frac{4\pi \cdot 1.33}{1000} + \frac{2\pi \cdot 1.34}{500}\right) = \pi$

$$z = 93\ nm.$$

h. The coherence length in a transmission experiment is much longer than in a reflection experiment which means that the amount of SH photons will be much larger than for transmission than for reflection,



but here there are other effects such as diffraction / scattering and absorption.

The Manley Rowe relations

Now we turn to the conservation of energy (intensity).

The intensity is defined as:

$$I_i = 2n_i \epsilon_0 c A_i A_i^*$$

the change in I_i $\frac{dI_i}{dz}$: $\frac{dI_i}{dz} = 2n_i \epsilon_0 c \left\{ A_i^* \frac{dA_i}{dz} + A_i \frac{dA_i^*}{dz} \right\}$ insert $\frac{dA_i}{dz}$

$$\frac{dI_1}{dz} = \frac{2n_i \epsilon_0 c \chi^{(2)}}{k_1 c^2} \omega_1^2 \left(i A_1^* A_3 A_2^* e^{-i\Delta k z} + c.c. \right)$$

$k = \frac{n\omega}{c}$

Note: $\underbrace{ie^{-ix}}_{i(\cos x - i\sin x)} - ie^{x} = i(\cos x - i\sin x) - i(\cos x + i\sin x) = 2\sin x$
$= -2\,\mathrm{Im}\left(e^{-ix}\right)$

$$\frac{c\omega_1^2}{k_1 c^2} = \frac{c\omega_1^2}{c^2} \cdot \frac{c}{n\omega_1}$$

$$\frac{dI_1}{dz} = -4\epsilon_0 \chi^{(2)} \omega_1 \,\mathrm{Im}\left(A_3 A_1^* A_2^* e^{-i\Delta k z} \right)$$

$$\frac{dI_2}{dz} = -4\epsilon_0 \chi^{(2)} \omega_2 \,\mathrm{Im}\left(A_3 A_1^* A_2^* e^{-i\Delta k z} \right)$$

$$\frac{dI_3}{dz} = -4\epsilon_0 \chi^{(2)} \omega_3 \,\mathrm{Im}\left(A_3^* A_1 A_2 e^{i\Delta k z} \right) = 4\epsilon_0 \chi^{(2)} \omega_3 \,\mathrm{Im}\left(A_3 A_1^* A_2^* e^{-i\Delta k z} \right)$$

$I = I_1 + I_2 + I_3$ is the total intensity (energy per area per time)

$$\frac{dI}{dz} = \frac{dI_1}{dz} + \frac{dI_2}{dz} + \frac{dI_3}{dz} = -4\epsilon_0 \chi^{(2)} \,\mathrm{Im}\left(A_3^* A_1 A_2 e^{-i\Delta k z} \right) \cdot \underbrace{\left(\omega_1 + \omega_2 - \omega_3 \right)}_{=0}$$

Thus there is conservation of energy, but also:

$$\frac{d}{dz}\left(\frac{I_1}{\omega_1} \right) = \frac{d}{dz}\left(\frac{I_2}{\omega_2} \right) = -\frac{d}{dz}\left(\frac{I_3}{\omega_3} \right)$$

The ~~an~~ change in photons is coupled

$\frac{I_1}{\omega_1}$: $\frac{\text{photons}}{\text{area} \cdot \text{time}}$